Individual Round

Lexington High School

April 9, 2016

1. Find the ordered triple of natural numbers (x, y, z) such that $x \le y \le z$ and $x^x + y^y + z^z = 3382$.

2. Mike rides a bike for 30 minutes, traveling 8 miles. He started riding at 20 miles per hour, but by the end of his journey he was only traveling at 10 miles per hour. What was his average speed, in miles per hour?

3. The squares of two positive integers differ by 2016. Find their maximum possible sum.

4. A triangle has two sides of lengths 1984 and 2016. Find the maximum possible area of the triangle.

5. An isosceles triangle has angles of 50°, x°, and y°. Find the maximum possible value of $x - y$.

6. A positive integer is called *cool* if it can be expressed in the form $a! \cdot b! + 315$ where a, b are positive integers. For example, $1! \cdot 1! + 315 = 316$ is a cool number. Find the sum of all cool numbers that are also prime numbers.

7. Compute the product of the three smallest prime factors of

$$21! \cdot 14! + 21! \cdot 21 + 14! \cdot 14 + 21 \cdot 14.$$

8. How many lattice points P in or on the circle $x^2 + y^2 = 25$ have the property that there exists a unique line with rational slope through P that divides the circle into two parts with equal areas?

9. An acute triangle has area 84 and perimeter 42, with each side being at least 10 units long. Let S be the set of points that are within 5 units of some vertex of the triangle. What fraction of the area of S lies outside the triangle?

10. There are sixteen buildings all on the same side of a street. How many ways can we choose a nonempty subset of the buildings such that there is an odd number of buildings between each pair of buildings in the subset?

11. Find all ordered triples (a, b, c) of real numbers such that

$$\begin{cases} a + b = c, \\ a^2 + b^2 = c^2 - c - 6, \\ a^3 + b^3 = c^3 - 2c^2 - 5c. \end{cases}$$

12. Two lines intersect inside a unit square, splitting it into four regions. Find the maximum product of the areas of the four regions.

13. Find the area enclosed by the graph of $|x| + |2y| = 12$.

14. Let P and Q be points on AC and AB, respectively, of triangle $\triangle ABC$ such that $PB = PC$ and $PQ \perp AB$. Suppose $\frac{AQ}{QB} = \frac{AP}{PB}$. Find $\angle CBA$, in degrees.

15. For nonnegative integers n, let $f(n)$ be the number of digits of n that are at least 5. Let $g(n) = 3^{f(n)}$. Compute

$$\sum_{i=1}^{1000} g(i).$$

16. Let N be the number of functions $f : \{1,2,3,4,5,6,7,8,9,10\} \to \{1,2,3,4,5\}$ that have the property that for $1 \le x \le 5$ it is true that $f(f(x)) = x$. Given that N can be written in the form $5^a \cdot b$ for positive integers a and b with b not divisible by 5, find $a+b$.

17. Find the minimum possible value of

$$\left\lfloor \frac{a+b}{c} \right\rfloor + 2\left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$$

where a, b, c are the sidelengths of a triangle.

18. Let $\triangle ABC$ be a triangle with $AB = 5, BC = 6, CA = 7$. Suppose P is a point inside $\triangle ABC$ such that $\triangle BPA \sim \triangle APC$. If AP intersects BC at X, find $\frac{BX}{CX}$.

19. Find the shortest distance between the graphs of $y = x^2 + 5$ and $x = y^2 + 5$.

20. Find the number of partitions of the set $\{1,2,3,\cdots,11,12\}$ into three nonempty subsets such that no subset has two elements which differ by 1.

21. Let S be the set of positive integers n such that

$$3 \cdot \varphi(n) = n,$$

where $\varphi(n)$ is the number of positive integers $k \le n$ such that $\gcd(k,n) = 1$. Find

$$\sum_{n \in S} \frac{1}{n}.$$

22. Albert rolls a fair six-sided die thirteen times. For each time he rolls a number that is strictly greater than the previous number he rolled, he gains a point, where his first roll does not gain him a point. Find the expected number of points that Albert receives.

23. Call a positive integer $n \ge 2$ *junk* if there exist two distinct n digit binary strings $a_1a_2\cdots a_n$ and $b_1b_2\cdots b_n$ such that

- $a_1 + a_2 = b_1 + b_2$,
- $a_{i-1} + a_i + a_{i+1} = b_{i-1} + b_i + b_{i+1}$ for all $2 \le i \le n-1$, and
- $a_{n-1} + a_n = b_{n-1} + b_n$.

Find the number of junk positive integers less than or equal to 2016.

24. Let S be a set consisting of all positive integers less than or equal to 100. Let P be a subset of S such that there do not exist two elements $x, y \in P$ such that $x = 2y$. Find the maximum possible number of elements of P.

25. Let $ABCD$ be a trapezoid with $AB \parallel DC$. Let M be the midpoint of CD. If $AD \perp CD$, $AC \perp BM$, and $BC \perp BD$, find $\frac{AB}{CD}$.